Chapter Five
Identity and Operation Symbols

1 IDENTITY

A certain relation is given a special treatment in logic. This is the *identity* relation -- the relation that relates each thing to itself and relates no thing to another thing. It is represented by a two-place predicate. For historical reasons, it is usually written as the *equals* sign of arithmetic, and instead of being written in the position that we use for other predicates, in front of its terms:

=(xy)

it is written in between its terms:

x=y

Except for its special shape and location, it is just like any other two-place predicate. So the following are formulas:

a=x
b=z ∨ ~b=c
Ax → x=x
∀x∀y[x=a → [a=y → x=y]]
∀x[Bx → ∃y[Cy ∧ x=y]]

This sign is used to symbolize the word 'is' in English when that word is used between two names. For example, according to the famous story, Dr. Jekyll is Mr. Hyde, so using 'e' for Jekyll and 'h' for Hyde we write 'Jekyll is Hyde' as 'e=h'. And using 'c' for 'Clark Kent', 'a' for 'Superman', and 'd' for Jimmy Olsen we can write:

a=c ∧ ~a=d *Superman is Clark Kent but Superman is not Jimmy Olsen*

It is customary to abbreviate the negation of an identity formula by writing a slash through the identity sign: '≠' instead of putting the negation sign in front. So we could write:

a=c ∧ a≠d *Superman is Clark Kent but Superman is not Jimmy Olsen*

We can represent the following argument:

Superman is either Clark Kent or Jimmy Olsen
Superman is not Jimmy Olsen
∴ *Superman is Clark Kent*

as:

a=c ∨ a=j
a≠j
∴ a=c

with the short derivation:

1. ~~Show~~ a=c
2. | a=c pr1 pr2 mtp dd |

There are other ways of saying 'is'. The word 'same' sometimes conveys the sense of identity -- and sometimes not. Consider the claim:

Bozo and Herbie were wearing the same pants.

This could simply mean that they were wearing pants of the same style; if so, that is not identity in the logical sense. But it could mean that there was a single pair of pants that they were both inside of; that would mean identity.

The word 'other' is often meant as the negation of identity. In the following sentences:

> *Agatha saw a dragonfly and Betty saw a dragonfly*
> *Agatha saw a dragonfly and Betty saw another dragonfly*

the first sentence is neutral about whether they saw the same dragonfly, but in the second sentence Betty saw a dragonfly that was not the same dragonfly that Agatha saw:

$\exists x[Dx \wedge S(ax)] \wedge \exists y[Dy \wedge S(by)]$
$\exists x[Dx \wedge S(ax) \wedge \exists y[Dy \wedge y \neq x \wedge S(by)]]$

↖ *y is other than x*

EXERCISES

1. Say which of the following are formulas:

a. $Fa \wedge Gb \wedge F=G$
b. $\forall x \forall y[R(xy) \rightarrow x=y]$
c. $\forall x \forall y[R(xy) \wedge x \neq y \leftrightarrow S(yx)]$
d. $R(xy) \wedge R(yx) \leftrightarrow x=y$
e. $\exists x \exists y[x=y \wedge y \neq x]$

2. Symbolize the following English sentences:

a. Bruce Wayne is Batman
b. Bruce Wayne isn't Superman
c. If Clark Kent is Superman, Clark Kent is not from Earth
d. If Clark Kent is Superman, Superman is a reporter
e. Felecia chased a dog and Cecelia chased a dog.
f. Felecia chased a dog and Cecelia chased the same dog.
g. Felecia chased a dog and Cecelia chased a different dog.

2 AT LEAST, AT MOST, EXACTLY, AND ONLY

The use of the identity predicate lets us express certain complex relations using logical notation.

At least one: If we want to say that Betty saw at least one dragonfly we can just write that there is a dragonfly that she saw:

$$\exists x[Dx \wedge S(bx)]$$

At least two: If we want to say that Betty saw at least two dragonflies, we can say that she saw a dragonfly and she saw *another* dragonfly, i.e. a dragonfly that wasn't the first dragonfly:

$$\exists x[x \textit{ is a dragonfly that Betty saw} \wedge \exists y[y \textit{ is a dragonfly other than x that Betty saw}]]$$
$$\exists x[Dx \wedge S(bx) \wedge \exists y[Dy \wedge y{\neq}x \wedge S(by)]]$$

This makes use of a negation of the identity predicate to symbolize 'another'.
The position of the second quantifier is not crucial; we could also write the slightly simpler formula:

$$\exists x\exists y[Dx \wedge S(bx) \wedge Dy \wedge S(by) \wedge y{\neq}x]$$

The non-identity in the last conjunct is essential; without it the sentence just gives the information that Betty saw a dragonfly and Betty saw a dragonfly without saying whether it was the same one or not.

At least three: If we want to say that Betty saw at least three dragonflies we can say that she saw a dragonfly, and she saw another dragonfly, and she saw yet another dragonfly -- i.e. a dragonfly that is not the same as either the first or the second:

$$\exists x[Dx \wedge S(bx) \wedge \exists y[Dy \wedge y{\neq}x \wedge S(by) \wedge \exists z[Dz \wedge z{\neq}x \wedge z{\neq}y \wedge S(bz)]]]$$

Again, the quantifiers may all occur in initial position:

$$\exists x\exists y\exists z[y{\neq}x \wedge z{\neq}x \wedge z{\neq}y \wedge Dx \wedge S(bx) \wedge Dy \wedge S(by) \wedge Dz \wedge S(bz)]$$

At most one: If we want to say that Betty saw at most one dragonfly, we can say that if she saw a dragonfly and a dragonfly, they were the same:

$$\forall x\forall y[x \text{ is a dragonfly that Betty saw} \wedge y \text{ is a dragonfly that Betty saw} \rightarrow x{=}y]$$
$$\forall x\forall y[Dx \wedge Dy \wedge S(bx) \wedge S(by) \rightarrow x{=}y]$$

This doesn't say whether Betty saw any dragonflies at all; it merely requires that she didn't see more than one. We can also symbolize this by saying that she didn't see at least two dragonflies:

$$\sim\exists x\exists y[Dx \wedge Dy \wedge S(bx) \wedge S(by) \wedge y{\neq}x]$$

It is easy to show that these two symbolizations are equivalent:

1. ~~Show~~ $\forall x\forall y[Dx \wedge Dy \wedge S(bx) \wedge S(by) \rightarrow x{=}y] \leftrightarrow \sim\exists x\exists y[Dx \wedge Dy \wedge S(bx) \wedge S(by) \wedge y{\neq}x]$

2.	$\sim\exists x\exists y[Dx \wedge Dy \wedge S(bx) \wedge S(by) \wedge y{\neq}x]$	ass bd
3.	$\forall x\forall y\sim[Dx \wedge Dy \wedge S(bx) \wedge S(by) \wedge y{\neq}x]$	2 ie/qn ie/qn
4.	$\forall x\forall y[Dx \wedge Dy \wedge S(bx) \wedge S(by) \rightarrow x{=}y]$	3 ie/nc bd

At most two: If we want to say that Betty saw at most two dragonflies either of the above styles will do:

$$\forall x\forall y\forall z[Dx \wedge Dy \wedge Dz \wedge S(bx) \wedge S(by) \wedge S(bz) \rightarrow x{=}y \vee x{=}z \vee y{=}z]$$
$$\sim\exists x\exists y\exists z[Dx \wedge Dy \wedge Dz \wedge x{\neq}y \wedge y{\neq}z \wedge x{\neq}z \wedge S(bx) \wedge S(by) \wedge S(bz)]$$

Exactly one: There are two natural ways to say that Betty saw exactly one dragonfly. One is to conjoin the claims that she saw at least one and that she saw at most one:

$$\exists x[Dx \wedge S(bx)] \wedge \forall x\forall y[Dx \wedge Dy \wedge S(bx) \wedge S(by) \rightarrow x{=}y]$$

Or we can say that she saw a dragonfly, and any dragonfly she saw was that one:

$$\exists x[Dx \wedge S(bx) \wedge \forall y[Dy \wedge S(by) \rightarrow x{=}y]]$$

Or, even more briefly:

$$\exists x\forall y[Dy \wedge S(by) \leftrightarrow y{=}x]$$

Exactly two: Similarly with exactly two; we can use the conjunction of she saw at least two and she saw at most two:

$$\exists x\exists y[Dx \wedge S(bx) \wedge Dy \wedge S(by) \wedge y{\neq}x] \wedge \forall x\forall y\forall z[Dx \wedge Dy \wedge Dz \wedge S(bx) \wedge S(by) \wedge S(bz) \rightarrow x{=}y \vee x{=}z \vee y{=}z]$$

or we can say that she saw two dragonflies, and any dragonfly she saw is one of them:

$$\exists x\exists y[Dx \wedge S(bx) \wedge Dy \wedge S(by) \wedge y{\neq}x \wedge \forall z[Dz \wedge S(bz) \rightarrow x{=}z \vee y{=}z]]$$

or, even more briefly:

$$\exists x\exists y[y{\neq}x \wedge \forall z[Dz \wedge S(bz) \leftrightarrow z{=}x \vee z{=}y]]$$

Talk of at least, or at most, or exactly, frequently occurs within larger contexts. For example:

Some giraffe that saw at least two hyenas was seen by at most two lions

∃x[x is a giraffe ∧ x saw at least two hyenas ∧
x was seen by at most two lions]

i.e.

$$\exists x[Gx \wedge \exists y\exists z[Hy \wedge Hz \wedge y{\neq}z \wedge S(xy) \wedge S(xz)] \wedge \forall u\forall v\forall w[Lu \wedge Lv \wedge Lw \wedge S(ux) \wedge S(vx) \wedge S(wx) \rightarrow u{=}v \vee v{=}w \vee u{=}w]]$$

Or this:

Each giraffe that saw exactly one hyena saw a lion that exactly one hyena saw

∀x[x is a giraffe ∧ x saw exactly one hyena → ∃y[Ly ∧ exactly one hyena saw y ∧ x saw y]]

$$\forall x[Gx \wedge \exists z[Hz \wedge S(xz) \wedge \forall u[Hu \wedge S(xu) \rightarrow u{=}z]] \rightarrow \exists y[Ly \wedge \exists v[Hv \wedge S(vy) \wedge \forall w[Hw \wedge S(wy) \rightarrow w{=}v]] \wedge S(xy)]]$$

Only: In chapter 1 we saw how to symbolize claims with '*only if*', and in chapter 3 we discussed how to symbolize '*only As are Bs*'. When '*only*' occurs with a name, it has a similar symbolization. Saying that *only giraffes are happy* is to say that anything that is happy is a giraffe:

$$\forall x[Hx \rightarrow Gx]$$

or that nothing that isn't a giraffe is happy:

$$\sim\exists x[\sim Gx \wedge Hx]$$

With a name or variable the use of 'only' is generally taken to express a stronger claim. For example, '*only Cynthia sees Dorothy*' is generally taken to imply that Cynthia sees Dorothy, and that anyone who sees Dorothy *is* Cynthia:

$$S(cd) \wedge \forall x[S(xd) \rightarrow x{=}c]$$

This can be symbolized briefly as:

$$\forall x[S(xd) \leftrightarrow x{=}c]$$

We have seen that '*another*' can often be represented by the negation of an identity; the same is true of '*except*' and '*different*':

No freshman except Betty is happy.
~∃x[x is a freshman ∧ x isn't Betty ∧ x is happy]
$\sim\exists x[Fx \wedge \sim x = b \wedge Hx]$

This has the same meaning as '*No freshman besides Betty is happy*'. Notice that neither of these sentences entail that Betty is happy. That is because one could reasonably say something like '*No freshman except Betty is happy, and for all I know she isn't happy either*'. So the sentence by itself does not say that Betty herself is happy, although if you knew that the speaker knew whether or not Betty is happy then since the speaker didn't say '*No freshman is happy*', you can assume that the speaker thinks Betty is happy.

Lastly:

Betty groomed a dog and Cynthia groomed a different dog.

∃x[x *is a dog* ∧ *Betty groomed* x ∧ ∃y[y *is a dog* ∧ y *is different from* x ∧ *Cynthia groomed* y]]

∃x[Dx∧ G(bx) ∧ ∃y[Dy ∧ ~y = x ∧ G(cy)]]

EXERCISES

1. Symbolize each of the following,

   a. *At most one candidate will win at least two elections*
   b. *Exactly one election will be won by no candidate*
   c*. Betty saw at least two hyenas which (each) saw at most one giraffe.*

2. The text states that one can symbolize '*Betty saw exactly one dragonfly*' as:

   ∃x∀y[Dy ∧ S(by) ↔ y=x].

Prove that this sentence is equivalent to one of the other symbolizations given in the text for '*exactly one*'.

3. Similarly show that one can symbolize '*Betty saw exactly two dragonflies*' as:

   ∃x∃y[x≠y ∧ ∀z[Dz ∧ S(bz) ↔ z=x ∨ z=y]]

by showing that this is equivalent to one of the other symbolizations given in the text.

4. Show that the two symbolizations proposed above for *only Cynthia sees Dorothy* are equivalent:

   ∴ S(cd) ∧ ∀x[S(xd) → x=c] ↔ ∀x[S(xd) ↔ x=c]

3 DERIVATIONAL RULES FOR IDENTITY

Identity brings along with it two fundamental logical rules. One stems from the principle that everything is identical to itself. This rule, called "self-identity", allows one to write a self-identity on any line of any derivation:

> **Rule sid ("self-identity")**
>
> On any line one may write two occurrences of the same term flanking the identity sign.
> As justification write "sid".

This rule is not often used, but when it is needed, it is straightforward. For example, it can be used to show that this argument is valid:

$\forall x\ x=x \rightarrow P$
$\therefore\ P$

1.	~~Show~~ P		
2.	$\sim P$	ass id	
3.	$\sim\forall x\ x=x$	2 pr1 mt	
4.	$\exists x \sim x=x$	3 qn	
5.	$\sim u=u$	4 ei	
6.	$u=u$	sid	← Rule sid
7.		5 6 id	

Or, more briefly:

1.	~~Show~~ P		
2.	~~Show~~ $\forall x\ x=x$		
3.	$x=x$	sid ud	← Rule sid
4.	P	pr1 3 mp dd	

The more commonly used rule is called Leibniz's Law, for the 17-18th century philosopher Gottfried Wilhelm von Leibniz. It is an application of the principle that if x=y then whatever is true of x is true of y. Specifically:

> **Rule LL ("Leibniz's Law")**
>
> If a formula of the form 'a=b' (or 'b=a') occurs on an available line, and if a formula containing 'a' also occurs on an available line, then one may write the same formula with any number of free occurrences of 'a' changed to free occurrences of 'b'. As justification, write the line numbers of the earlier lines along with 'LL'.
>
> This rule applies whether 'a' and 'b' are variables or names (or complex terms -- to be introduced below). (Occurrence of names are automatically considered free.)

Example:

	Cynthia saw a rabbit, and nothing else.	$\exists x[Rx \wedge S(cx) \wedge \forall y[S(cy) \rightarrow \sim y\neq x]]$
	Cynthia saw Henry	$S(ch)$
$\therefore$	*Henry is a rabbit*	Rh

1.	~~Show~~ Rh	
2.	Ru ∧ S(cu) ∧ ∀y[S(cy) → ~y≠u]	pr1 ei
3.	∀y[S(cy) → ~y≠u]	2 s
4.	S(ch) → ~h≠u	3 ui
5.	~h≠u	pr2 4 mp
6.	h=u	5 dn
7.	Ru	2 s s
8.	Rh	6 7 LL dd

It is convenient to also have a contrapositive form of Leibniz's law, saying that if something that is true of a is not true of b, then a≠b. For example:

Fa ∧ S(ac)
~[Fb ∧ S(bc)]
∴ a≠b

This inference is easily attainable with an indirect derivation: assume 'a=b' and use LL with the premises to derive a contradiction. But it is convenient to include this as a special case of Leibniz's law itself:

Rule LL (contrapositive form)

The formula 'a≠b' may be written on a line if a formula containing 'a' occurs on an available line, and if the negation of that same formula occurs on another available line with any number of free occurrences of 'a' changed to free occurrences of 'b'. As justification, write the line numbers of the earlier lines along with 'LL'.

This rule applies whether 'a' and 'b' are variables or names (or complex terms -- to be introduced below). (Occurrences of names are automatically considered free.)

An additional rule is derivable from the rules at hand. It is called Symmetry because it says that identity is symmetric: if x=y then y=x:

Rule sm (symmetry)

If an identity formula (or the negation of an identity formula) occurs on an available line or premise, one may write that formula with its left and right terms interchanged.

As justification, write the earlier line number and 'sm'.

Examples of derivations using this rule are:

∃x[x=b ∧ Fx]
∀x[b=x → Gx]
∴ ∃x[Fx ∧ Gx]

1.	~~Show~~ ∃x[Fx ∧ Gx]		
2.	u=b ∧ Fu	pr1 ei	
3.	u=b	2 s	
4.	b=u → Gb	pr2 ui	
5.	b=u	3 sm	← rule sm
6.	Gb	4 5 mp	
7.	Fu	2 s	
8.	Fb	3 7 LL	
9.	Fb ∧ Gb	6 8 adj	
10.	∃x[Fx ∧ Gx]	9 eg dd	

∴ $\forall x[x=a \rightarrow a=x]$

1. ~~Show~~ $\forall x[x=a \rightarrow a=x]$
2. ~~Show~~ $x=a \rightarrow a=x$
3. $x=a$ — ass cd
4. $a=x$ — 3 sm cd ← rule sm
5. 2 ud

EXERCISES

1. Produce derivations for the following theorems:

T301	$\forall x\ x=x$	identity is "reflexive"
T302	$\forall x \forall y[x=y \leftrightarrow y=x]$	identity is "symmetric"
T303	$\forall x \forall y \forall z[x=y \wedge y=z \rightarrow x=z]$	identity is "transitive"
T304	$\forall x \forall y[x=y \rightarrow [Fx \leftrightarrow Fy]]$	
T306	$\forall x[Fx \leftrightarrow \forall y[y=x \rightarrow Fy]]$	
T307	$\forall x[Fx \leftrightarrow \exists y[y=x \wedge Fy]]$	
T322	$\exists x \forall y\ y=x \leftrightarrow \forall x \forall y\ y=x$	
T323	$\exists x \exists y\ x \neq y \leftrightarrow \forall x \exists y\ x \neq y$	
T329	$\forall y \exists x\ x=y$	
T330	$\forall y \exists z \forall x[x=y \leftrightarrow x=z]$	

2. Produce derivations for the following valid arguments.

a. $\forall x[Fx \rightarrow x=a \vee x=b]$
$\sim Fa$
$\sim Gb$
∴ $\forall x[Fx \rightarrow \sim Gx]$

b. $\exists x \forall y[Ay \leftrightarrow y=x]$
∴ $\exists x[Ax \wedge \sim Bx] \leftrightarrow \sim \exists x[Ax \wedge Bx]$

c. $\exists x \exists y[x \neq y \wedge Gx \wedge Gy]$
$\forall x[Gx \rightarrow Hx]$
∴ $\sim \exists x \forall y[Hy \leftrightarrow y=x]$

d. $\exists x \exists y[Fx \wedge Fy \wedge x \neq y]$
$\exists x \exists y[Gx \wedge Gy \wedge x \neq y]$
∴ $\exists x \exists y[Fx \wedge Gy \wedge x \neq y]$

3. Symbolize these arguments and produce derivations to show that they are valid.

a. Every giraffe that loves some other giraffe loves itself.
Every giraffe loves some giraffe.
∴ Every giraffe loves itself.

b. No cat that likes at least two dogs is happy.
Tabby is a cat that likes Fido.
Tabby likes a dog that Betty owns.
Fido is a dog.
Tabby is happy.
∴ Betty owns Fido.

c. Each widget fits into a socket.
Widget *a* doesn't fit into socket *f*
∴ Widget *a* fits into some socket other than *f*

d. Only Betty and Carl were eligible
Somebody who was eligible, won
Carl didn't win
∴ Betty won

4 INVALIDITIES WITH IDENTITY

The presence of the identity relation does not change our technique for showing invalidity. The only addition is the constraint that the identity predicate must have *identity* as its extension. That is, its extension must consist of all the ordered pairs whose first and second member are the same. So, if the universe is {0, 1, 2}, the extension of identity must be:

$=$: $\{\langle 0,0\rangle, \langle 1,1\rangle, \langle 2,2\rangle\}$

Since this is completely determined, it is customary to take this for granted, and not to bother stating an extension for the identity sign.

An example of an invalid argument involving identity is:

	$Fa \wedge Gb \wedge a\neq b$	*Andrews is fast and Betty is good, but Andrews isn't Betty*
	$Gb \wedge Fc \wedge b\neq c$	*Betty is good and Cynthia is fast, but Betty isn't Cynthia*
∴	$Fa \wedge Fc \wedge a\neq c$	*Andrews is fast and Cynthia is fast, but Andrews isn't Cynthia*

COUNTER-EXAMPLE:

Universe: {0, 1,}
a: 0
b: 1
c: 0
F: {0}
G: {1}

The first premise is true because 0 is F and 1 is G and $0\neq 1$. The second premise is true because 1 is G and 0 is F and $1\neq 0$. But the last conjunct of the conclusion is false, since 0 *is* 0.

As before, sometimes a counter-example requires an infinite universe. An example is this argument:

$\forall x[R(xc) \rightarrow x=c]$
$\forall x\exists y[R(xy) \wedge x\neq y]$
∴ $\sim\forall x\forall y\forall z[R(xy) \wedge R(yz) \rightarrow R(xz)]$

COUNTER-EXAMPLE

Universe: {0, 1, 2, . . . }
c: 0
R(①②): ①≤②

The first premise is true because the only thing in the given universe less than or equal to 0 is 0 itself. The second is true because for each thing there is something greater than it (and different from it). And the conclusion is false because ≤ is transitive.

EXERCISES

1. Only Betty and Carl were eligible
Nobody who wasn't eligible won
Carl didn't win
∴ Betty won

2. Ann loves at least one freshman.
Ann loves David.
Ed is a freshman.
David isn't Ed.
∴ There are at least two freshmen

3. Lois sees Clark at a time if and only if she sees Superman at that time.
∴ Clark is Superman

4. Gertrude sees at most one giraffe
Gertrude sees Fred, who is a giraffe
Bob is a giraffe
∴ Gertrude doesn't see Bob

5 OPERATION SYMBOLS

So far we have dealt only with simple terms: variables and names. In mathematics and in science complex terms are common. Some familiar examples from arithmetic are:

$-x$, x^2, $\sqrt{x}$, . . .	negative x, x squared, the square root of x
$x+y$, $x-y$, $x \times y$, . . .	x plus y, x minus y, x times y

These complex terms consist of variables combined with special symbols called *operation symbols*. The operation symbols on the first line are *one-place* operation symbols; they each combine with *one* variable to make a complex term. The *two-place* operation symbols on the second line each combine with *two* variables to make a complex term. Operation symbols also combine with names. It is customary in arithmetic to treat numerals as names of numbers. When numeral names combine with operation symbols we get complex signs such as:

-4, 7^2, $\sqrt{9}$, . . .
$4+7$, $21-13$, 5×8, . . .

Each of these is taken to be a complex term. For example, '-4' is a complex term standing for the number, negative four; '7^2' is a complex term standing for the number forty-nine; '5×8' is a complex term standing for the number forty, and so on.

In logical notation we use any small letter between 'a' and 'h' as an operation symbol; the terms that they combine with are enclosed in parentheses following them. So if 'a' stands for the squaring operation, we write 'a‹x›' for what is represented in arithmetic as 'x^2' and if 'b' stands for the addition operation, we write 'b‹xy›' for what is represented in arithmetic as 'x+y'. Specifically:

Terms
Simple names (the letters 'a' through 'h') and variables (the letters 'i' through 'z') are terms.
Any small letter between 'a' and 'h' can be used as an operation symbol.
Any operation symbol followed by some number of terms in parentheses is a term.

The same letters are used both for names and for operation symbols. (It is often held that names are themselves *zero*-place operation symbols; a name makes a term by combining with nothing at all.) You can tell quickly whether a small letter between 'a' and 'h' is being used as a name or as an operation symbol: if it is directly followed by a left parenthesis, it is being used as an operation symbol; otherwise it is being used as a name.

Examples of terms are: 'b', 'w', 'e‹x›', 'f‹by›', 'h‹zbx›'. Since an operation symbol may combine with any term, it may combine with complex terms. So 'f‹z e‹x››' is a term, which consists of the operation symbol 'f' followed by the two terms: 'z' and 'e‹x›'. Terms can be much more complex than this. Consider the arithmetical expression:

$a \times (b^2 + c^2)$

If 'd' stands for the multiplication operation, 'e' for addition, and 'f' for squaring, this will be expressed in logical notation as:

d‹a e‹f‹b›f‹c›››

In arithmetic, operation symbols can go in front of the terms they combine with (as with '-4'), or between the terms they combine with (as with '5×8'), or to-the-right-and-above the terms they combine with (as with '7^2'), and so on. The logical notation used here uniformly puts operation symbols in front of the terms that they combine with.

We are used to seeing arithmetical notation used in equations with the equals sign. If numerals are names of numbers, then the equals sign can be taken to mean identity, and we can use our logical identity sign -- which already looks exactly like the equals sign -- for the equals sign. For example, we can take the equation:

7+5 = 12

to say that the number that '7+5' stands for is exactly the same number that '12' stands for. The equation will appear in logical notation as:

e‹ab› = c

EXERCISES

Which of the following are formulas?

a. R(f‹x›g‹x›)
b. ∀x[Fx → Fg‹x›]
c. ∀x[Fx → Fg‹xx›]
d. ∀x∀y[x=h‹y› → f‹xy›=f‹yx›]
e. ~∃y∃x[x=f‹y› ∧ y=f‹x›]
f. ~∃x∃yf‹xy›
g. S(xyz) ∨ ~S(xg‹y›z) ∨ ~S(g‹xy›g‹z›g‹yz›)
h. Fa ∧ ~Fb → [Fg‹a› → g‹b›≠g‹a›]

6 DERIVATIONS WITH COMPLEX TERMS

Complex terms made with operation symbols do not require any additional rules of derivation. All that is needed is a clarification of previous rules regarding free occurrences of terms. Recall that if we are going to apply Leibniz's Law, there is a restriction that the occurrences of terms being changed be free ones. This is to forbid fallacious inferences like this one:

5.	x=a	<derived somehow>	
6.	∃x(Fx ∧ Gx)	<derived somehow>	
7.	∃x(Fa ∧ Gx)	5 6 LL	← incorrect step

This inference is prevented by the restriction on Leibniz's Law that says that both the term being replaced and its replacement be free occurrences at the location of replacement. The displayed inference violates this constraint because it replaces a bound occurrence of 'x' by 'a'. When Leibniz's Law is applied to complex terms we say that a complex term is considered not to be free if it contains any variables that are bound by a quantifier outside the term; otherwise it is free. So, for example, this is fallacious:

5.	h‹x›=a	<derived somehow>	
6.	∃x(Fh‹x› ∧ Gx)	<derived somehow>	
7.	∃x(Fa ∧ Gx)	5 6 LL	← incorrect step

This application of Leibniz's Law is incorrect since the occurrence of the term 'h‹x›' being replaced has its 'x' bound by a quantifier outside that term on line 6. The following is OK since no variable becomes bound:

5.	h‹y›=a	<derived somehow>
6.	∃x(Fh‹y› ∧ Gx)	<derived somehow>
7.	∃x(Fa ∧ Gx)	5 6 LL

Some arithmetical calculations with complex terms are just applications of the logic of identity. For example, given that 2+3=5, and that 5+2=7 we can prove by the logic of identity alone that 7=(2+3)+2. This inference has the form:

	e‹ab› = c	2+3 = 5
	e‹ca› = d	5+2 = 7
∴	d = e‹e‹ab›a›	7 = (2+3)+2

1.	~~Show~~ d = e‹e‹ab›a›		
2.	e‹e‹ab›a› = d	pr1 pr2 LL	<replacing 'c' in premise 2 by 'e‹ab›'
3.	d = e‹e‹ab›a›	2 sm dd	

Other similar inferences cannot be proved by logic alone. For example, we cannot prove '2+3=3+2' by logical principles alone, because the fact that the order of the terms flanking an addition sign doesn't matter is not a principle of logic. This pattern doesn't hold, for example, for subtraction; we don't have 2–3=3–2.

A simple consequence of our laws of identity is a principle that is sometimes called Euclid's Law, because it was used by the geometer Euclid. This law says that given an identity statement, you can infer another identity statement where both sides of the new identity differ only with respect to terms identified in the original identity statement. Some examples of Euclid's Law are:

$x=y$	$a=b$	$x=a^2$	$a^2=b^2$	$a+b=c+d$
∴ $x^2=y^2$	∴ $a+1=b+1$	∴ $3\times x=3\times a^2$	∴ $a^2+a^2=b^2+b^2$	∴ $(a+b)^2=(c+d)^2$

Euclid's Law (rule el)

From any identity statement one may infer another identity statement whose sides are the same except for having one or more free occurrences of one side of the original statement in place of one or more free occurrences of the other side of the original statement. As justification, cite the number of the available line plus 'el'.

This rule is only a convenience, since one can get along without it by combining the rule for self-identity with Leibniz's Law. For example, we can validate this use of Euclid's Law:

$a+b=c+d$
∴ $(a+b)^2=(c+d)^2$

with this derivation, which does not appeal to Euclid's Law:

1. ~~Show~~ $(a+b)^2=(c+d)^2$
2. | $(a+b)^2=(a+b)^2$ sid
3. | $(a+b)^2=(c+d)^2$ 2 pr1 LL dd

Mathematical equations often appear in the formulation of scientific principles. For example, in physics you might be given an equation saying that the force acting on a body is equal to the product of its mass times its acceleration. The scientific equation for this is typically written:

$F = ma$

From the point of view of our logical notation, this is a universal generalization of the form:

$\forall x[f\langle x\rangle = b\langle m\langle x\rangle a\langle x\rangle\rangle]$

where 'b' represents the operation symbol for multiplication, and where 'f‹x›' means "the force acting on x", 'm‹x›' means "the mass of x", and 'a‹x›' means "the acceleration of x produced by f‹x›".

Operation symbols are not common outside of mathematics and science. They are sometimes used in discussing kinship relations, where 'father of' and 'mother of' are treated as operation symbols. Here is a set of principles of biological kinship where:

'f‹x›' means *the father of x*
'e‹x›' means *the mother of x*
'Ax' means *x is male*
'Ex' means *x is female*
'I(xy)' means *x and y are (full) siblings*
'B(xy)' means x *is a brother of y*
'D(xy)' means *x is a daughter of y*

P1	$\forall x A f\langle x\rangle$	*Everyone's father is male*
P2	$\forall x E e\langle x\rangle$	*Everyone's mother is female*
P3	$\forall x\forall y[I(xy) \leftrightarrow x\neq y \wedge f\langle x\rangle=f\langle y\rangle \wedge e\langle x\rangle=e\langle y\rangle]$	*(Full) Siblings have the same mother and father*
P4	$\forall x\forall y[B(xy) \leftrightarrow Ax \wedge I(xy)]$	*A brother of someone is his/her male sibling*
P5	$\forall x\forall y[D(xy) \leftrightarrow Ex \wedge [y=f\langle x\rangle \vee y=e\langle x\rangle]]$	*A daughter of a person is a female such that that person is her father or her mother*
P6	$\forall x[Ax \leftrightarrow \sim Ex]$	*Someone is male if and only if that person is not female*

Some consequences of this theory are:

∴ ∀x∀y[I(xy) ∧ ∃z x=f‹z› → B(xy)]	*Any father who is someone's sibling is that person's brother*

1.	~~Show~~ ∀x∀y[I(xy) ∧ ∃z x=f‹z› → B(xy)]	
2.	~~Show~~ ∀y[I(xy) ∧ ∃z x=f‹z› → B(xy)]	
3.	~~Show~~ I(xy) ∧ ∃z x=f‹z› → B(xy)	
4.	I(xy) ∧ ∃z x=f‹z›	ass cd
5.	∃z x=f‹z›	4 s
6.	x =f‹u›	5 ei
7.	Af‹u›	pr1 ui
8.	Ax	6 7 LL
9.	I(xy)	4 s
10.	Ax ∧ I(xy)	8 9 adj
11.	B(xy) ↔ Ax ∧ I(xy)	pr4 ui ui
12.	B(xy)	11 bc 10 mp cd
13.		3 ud
14.		2 ud

∴ ~∃x f‹x›=e‹x›	*Nobody's father is that person's mother*

1.	~~Show~~ ~∃x f‹x›=e‹x›	
2.	∃x f‹x›=e‹x›	ass id
3.	f‹u›=e‹u›	2 ei
4.	Af‹u›	pr1 ui
5.	Ae‹u›	3 4 LL
6.	Ee‹u›	pr2 ui
7.	Ae‹u› ↔ ~Ee‹u›	pr6 ui
8.	~Ee‹u›	7 bc 5 mp 6 id

∴ c=a → f⟨e⟨c⟩⟩=f⟨e⟨a⟩⟩	*If Clark Kent is Superman, Clark's maternal grandfather is Superman's maternal grandfather* <a case of Euclid's Law>

1.	~~Show~~ c=a → f⟨e⟨c⟩⟩=f⟨e⟨a⟩⟩	
2.	c=a	ass cd
3.	f⟨e⟨c⟩⟩=f⟨e⟨a⟩⟩	2 el cd

Derivations with operation symbols can be hard to do; this happens often in mathematics, and it accounts for some of the reason that mathematics is thought to be difficult. An example of this is the typical development of the mathematical theory of *groups*. A group is a set of things which may be combined with a two-place operation symbolized by 'c'. In a group, each thing has an *inverse*; the inverse of a thing is represented using a one-place inverting operation symbol 'd'. And there is a neutral element 'e'. There are three axioms governing groups:

∀x∀y∀z c‹xc‹yz››=c‹c‹xy›z›	Combination is associative
∀x c‹xe›=x	Combining anything with e yields the original thing
∀x c‹xd‹x››=e	Combining anything with its inverse yields e

These axioms can be satisfied by a wide variety of structures. For example, the positive and negative integers together with zero satisfy these axioms when the method of combination is addition, the neutral element is 0, and the inverse of anything is its negative; in arithmetical notation the axioms look like:

∀x∀y∀z x+(y+z)=((x+y)+z)	Addition is associative
∀x x+0=x	Adding zero to anything yields that thing
∀x x+(–x)=0	Adding the negative of anything to that thing yields 0

A typical exercise in group theory is to show that the group axioms entail the "law of right-hand cancellation": the general principle whose arithmetical analogue is '∀x∀v∀u [x+u=v+u → x=v]':

∀x∀y∀z c‹xc‹yz››=c‹c‹xy›z›
∀x c‹xe›=x
∀x c‹xd‹x››=e
∴ ∀x∀v∀u [c‹xu›=c‹vu› → x=v]

[The reader should try to think up how to do this derivation before looking below.]

1.	~~Show~~ ∀x∀v∀u [c‹xu›=c‹vu› → x=v]	
2.	~~Show~~ c‹xu›=c‹vu› → x=v	
3.	c‹xu› = c‹vu›	ass cd
4.	c‹c‹xu›d‹u›› = c‹c‹vu›d‹u››	3 el
5.	c‹xc‹ud‹u››› = c‹c‹xu›d‹u››	pr1 ui/x ui/u ui/d‹u›
6.	c‹xc‹ud‹u››› = c‹c‹vu›d‹u››	4 5 LL
7.	c‹vc‹ud‹u››› = c‹c‹vu›d‹u››	pr1 ui/v ui/u ui/d‹u›
8.	c‹xc‹ud‹u››› = c‹vc‹ud‹u›››	6 7 LL
9.	c‹ud‹u›› = e	pr3 ui
10.	c‹xe› = c‹vc‹ud‹u›››	8 9 LL
11.	c‹xe› = c‹ve›	9 10 LL
12.	c‹xe›=x	pr2 ui
13.	x = c‹ve›	11 12 LL
14.	c‹ve›=v	pr2 ui
15.	x = v	13 14 LL cd
16.		2 ud ud ud

EXERCISES

1. Give derivations for these theorems:

a. ∴ ∀xR(xf‹x›) → ∀xR(e‹x›f‹e‹x››)
b. ∴ ∀x∀y[x=f‹y› ∧ y=f‹x› → f‹f‹x››=x]

2. Show that these are consequences of the theory of biological kinship given above.

a.~∃x[∃zB(xz) ∧ ∃zD(xz)] *No brother is a daughter*
b.~∃x[∃z x=f‹z› ∧ ∃z x=e‹z›] *No father is a mother*

3. Show that these are consequences of the axioms for groups given above.

a. ∀x∀y∀z[c‹xy›=c‹zy› → x=z] <proved above>
b. ∀x[∀yc‹yx›=y → x=e]
c. ∀x c‹xd‹x››=c‹d‹x›x›
d. ∀x∀y∀z[c‹yx›=c‹yz› → x=z]
e. ∀x∀y[c‹xy›=e → y=d‹x›]
f. ∀x d‹d‹x››=x

7 INVALID ARGUMENTS WITH OPERATION SYMBOLS

To show that an argument is formally invalid we give a counter-example: that is, we interpret its parts to get an argument with that form that has true premises and a false conclusion. To do that we need to say what the symbols stand for in the situation. We say what a name stands for by picking a member of the universe of the interpretation. We say what a monadic predicate stands for by saying which members of the universe are in its extension. We say what a two-place predicate stands for by saying which pairs of members of the universe are in its extension. And so on. We do something similar for operation symbols. We say what a one-place operation symbol 'f' stands for by saying for each thing in the universe what a complex name of the form 'f⟨a⟩' stands for when 'a' stands for that thing. We say what a two-place operation symbol 'f' stands for by saying for each pair of things in the universe what a complex name of the form 'f⟨ab⟩' stands for when 'a' stands for the first member of the pair and 'b' stands for the second member of the pair.

Consider the following invalid argument:

∀x[Ex ↔ Ef‹x›]
∃xEx
∃x~Ex
∴ ∃x x=f‹x›

The universe of our counter-example will consist of the numbers {0, 1, 2, 3}

We will interpret 'E' as meaning 'is even', so that its extension is given by:

E: {0,2}

This makes the second and third premises true. In order to make the first premise true we need the operation symbol 'f' to yield a name of an even number when it is combined with the name of an even number, and yield a non-even number when it is combined with the name of a non-even number. We cannot let f assign each thing to itself, because that would make the conclusion true. But we can do the following:

f‹0› = 2 f‹1› = 3 f‹2› = 0 f‹3› = 1

This means that when 'f' combines with a name, a, of 0, 'f‹a›' stands for 2, that when 'f' combines with a name, a, of 1, 'f‹a›' stands for 3, that when 'f' combines with a name, a, of 2, 'f‹a›' stands for 0, and when 'f' combines with a name, a, of 3, 'f‹a›' stands for 1. As a result the first premise is true, while the conclusion is false, and we have our desired counter-example.

Notice that in explaining f we must give an entry for each member of the universe, showing what that member produces when acted on by the operation f. This is because of our assumption that every name actually stands for something. This assumption is vital to the validity of rules eg and ui. If there were something in the universe which f did not operate on, then if 'a' were to name that thing, 'f‹a›' would be a name that did not stand for anything, and as a result we could not apply rule eg to a premise of the form 'Af‹a›'. Since we want rules such as eg to apply to any term, we have to insure that every term stands for something, and in the context of counter-examples that requires that every operation symbol is defined for every member of the universe.

Two-place operation symbols are treated the same, except that they require us to assign things to *pairs* of members of the universe. Consider this invalid argument:

∀x∃yf‹xy›=x
∀x∃yf‹xy›≠x
∴ ∃x∃y[x≠y ∧ f‹xy›=f‹yx›]

This can be shown invalid with a counter-example having a two-membered universe: {0,1}
We give the following for f:

f‹00› = 0 f‹01› = 1 f‹10› = 0 f‹11› = 1

The first premise is true because if 'x' is chosen to be 0, 'y' can be chosen to be 0, and if 'x' is chosen to be 1, 'y' can be chosen to be 1. The opposite choices make the second premise true. But the conclusion is false, because when 'x' and 'y' are different, their order makes a difference for 'f'.

If these judgments about the truth and falsity of the sentences in this counter-example are difficult, we can employ the method of truth-functional expansions from chapter 3. We introduce names for the members of the universe:

$\mathbf{i}_0$: 0
$\mathbf{i}_1$: 1

The first premise has as a partial expansion the conjunction:

pr1: $\exists y f\langle \mathbf{i}_0 y\rangle = \mathbf{i}_0 \wedge \exists y f\langle \mathbf{i}_1 y\rangle = \mathbf{i}_1$

Its full expansion results from expanding each conjunct into a disjunction:

pr1: $[f\langle \mathbf{i}_0\mathbf{i}_0\rangle = \mathbf{i}_0 \vee f\langle \mathbf{i}_0\mathbf{i}_1\rangle = \mathbf{i}_0] \wedge [f\langle \mathbf{i}_1\mathbf{i}_0\rangle = \mathbf{i}_1 \vee f\langle \mathbf{i}_1\mathbf{i}_1\rangle = \mathbf{i}_1]$

$$\underbrace{\underbrace{\mathrm{T}\quad\mathrm{F}}_{\mathrm{T}}\qquad\underbrace{\mathrm{F}\quad\mathrm{T}}_{\mathrm{T}}}_{\mathrm{T}}$$

So the first premise has a true expansion. The second premise has as a partial expansion:

pr2: $\exists y f\langle \mathbf{i}_0 y\rangle \neq \mathbf{i}_0 \wedge \exists y f\langle \mathbf{i}_1 y\rangle \neq \mathbf{i}_1$

and as a full expansion:

pr2: $[f\langle \mathbf{i}_0\mathbf{i}_0\rangle \neq \mathbf{i}_0 \vee f\langle \mathbf{i}_0\mathbf{i}_1\rangle \neq \mathbf{i}_0] \wedge [f\langle \mathbf{i}_1\mathbf{i}_0\rangle \neq \mathbf{i}_1 \vee f\langle \mathbf{i}_1\mathbf{i}_1\rangle \neq \mathbf{i}_1]$

$$\underbrace{\underbrace{\mathrm{F}\quad\mathrm{T}}_{\mathrm{T}}\qquad\underbrace{\mathrm{T}\quad\mathrm{F}}_{\mathrm{T}}}_{\mathrm{T}}$$

It, too, has a true expansion. The conclusion has as a partial expansion:

c: $\exists y[\mathbf{i}_0 \neq y \wedge f\langle \mathbf{i}_0 y\rangle = f\langle y\mathbf{i}_0\rangle] \vee \exists y[\mathbf{i}_1 \neq y \wedge f\langle \mathbf{i}_1 y\rangle = f\langle y\mathbf{i}_1\rangle]$

and as a full expansion:

$$[\mathbf{i}_0 \neq \mathbf{i}_0 \wedge f\langle \mathbf{i}_0\mathbf{i}_0\rangle = f\langle \mathbf{i}_0\mathbf{i}_0\rangle] \vee [\mathbf{i}_0 \neq \mathbf{i}_1 \wedge f\langle \mathbf{i}_0\mathbf{i}_1\rangle = f\langle \mathbf{i}_1\mathbf{i}_0\rangle] \vee [\mathbf{i}_1 \neq \mathbf{i}_0 \wedge f\langle \mathbf{i}_1\mathbf{i}_0\rangle = f\langle \mathbf{i}_0\mathbf{i}_1\rangle] \vee [\mathbf{i}_1 \neq \mathbf{i}_1 \wedge f\langle \mathbf{i}_1\mathbf{i}_1\rangle = f\langle \mathbf{i}_1\mathbf{i}_1\rangle]$$

$$\underbrace{\underbrace{\mathrm{F}\quad\mathrm{T}}_{\mathrm{F}}\qquad\underbrace{\mathrm{T}\quad\mathrm{F}}_{\mathrm{F}}\qquad\underbrace{\mathrm{T}\quad\mathrm{F}}_{\mathrm{F}}\qquad\underbrace{\mathrm{F}\quad\mathrm{T}}_{\mathrm{F}}}_{\mathrm{F}}$$

As usual, it is a bit complicated to produce these expansions, but easy to check them for truth-value once they are produced.

EXERCISES

1. Produce counter-examples to show that these arguments are not formally valid:

a. $\forall x\exists y\ a\langle xy\rangle=c$
$\forall x\exists y\ a\langle yx\rangle=c$
$\therefore\ \forall x\forall y\ a\langle xy\rangle=a\langle yx\rangle$

b. $\forall x\exists y\ a\langle xy\rangle=c$
$\forall x\exists y\ a\langle xy\rangle=d$
$\therefore\ \exists x\forall y\ a\langle xc\rangle=y$

c. $\forall x\forall y\ a\langle xy\rangle=a\langle yx\rangle$
$\therefore\ \forall z\exists x\exists y\ a\langle xy\rangle=z$

2. Show that these are not theorems of the theory of biological kinship given in the previous section:

a. $\forall x[\exists y\ x=f\langle y\rangle \vee \exists y\ x=e\langle y\rangle]$ *Everyone is a father or a mother*

b. $\sim\exists x\ x=e\langle x\rangle$ *Nobody is their own mother*

8 COUNTER-EXAMPLES WITH INFINITE UNIVERSES

Some invalid arguments with operation symbols need infinite universes for a counter-example. Here is an example:

$\forall x H(xg\langle x\rangle)$
$\forall x\forall y\forall z[H(xy) \wedge H(yz) \rightarrow H(xz)]$
$\therefore\ \exists x H(xx)$

A natural arithmetic counter-example is given by making the universe the non-negative integers {0,1,2, . .}, making 'g' stand for the successor operation, that is, the operation which associates with each number the number after it, and 'H' for the two-place relation of 'less than':

$g\langle \textcircled{1}\rangle$: $\textcircled{1}+1$ — whatever you apply g to, you get that thing plus 1
$H(\textcircled{1}\textcircled{2})$: $\textcircled{1}<\textcircled{2}$ — H relates two things iff the first is less than the second

The first premise then says that every integer is less than the integer you get by adding one to it, and the second says, as earlier, that less than is transitive. The conclusion falsely says that some integer is less then itself.

Sometimes an infinite universe is not required but it is convenient if a counter-example with an infinite universe springs to mind. That might happen with this argument:

$\forall x\forall y\ e\langle xy\rangle=e\langle yx\rangle$
$\forall x\ e\langle xa\rangle=x$
$\exists x\exists y\ f\langle xy\rangle\neq f\langle yx\rangle$
$\forall x\ f\langle xa\rangle=x$
$\therefore\ \exists x[x\neq a \wedge e\langle xx\rangle=x]$

Take as the universe all integers, positive, negative, and zero. Then interpret the symbols as follows:

a: 0
$e\langle \textcircled{1}\textcircled{2}\rangle$: $\textcircled{1}+\textcircled{2}$
$f\langle \textcircled{1}\textcircled{2}\rangle$: $\textcircled{1}-\textcircled{2}$

On this interpretation, the first premise says that ①+② is always the same as ②+①. The second says that ①+0=① for any integer ①. The third says that for some integers ① and ②, ①–② is not the same as ②–①. The fourth says that ①–0=① for any integer ①. The conclusion says falsely that for some integer other than 0, adding it to itself yields itself.

An infinite universe was not forced on us in this case. We could instead have taken as our universe the numbers {0,1}, and interpreted as follows:

a: 0

$e\langle 00\rangle = 0$	$e\langle 01\rangle = 1$	$e\langle 10\rangle = 1$	$e\langle 11\rangle = 0$
$f\langle 00\rangle = 0$	$f\langle 01\rangle = 0$	$f\langle 10\rangle = 1$	$f\langle 11\rangle = 0$

These choices make all the premises true and the conclusion false.

A very simple invalid argument that requires an infinite universe to show its invalidity is the following.

$\forall x\forall y[g\langle x\rangle=g\langle y\rangle \to x=y]$
$\therefore\ \exists x\, g\langle x\rangle = a$

Universe: {0, 1, 2, . . . }
g‹①›: ①+1
a: 0

EXERCISES

1. Show that this argument is invalid:

$\forall x\forall y\forall z[R(xy)\wedge R(yz) \to R(xz)]$
$\forall x R(xf\langle x\rangle)$
$\therefore\ \exists x R(xa)$

2. Show that the third axiom for groups does not follow from the first two axioms; i.e. that this is invalid:

$\forall x\forall y\forall z\ c\langle xc\langle yz\rangle\rangle=c\langle c\langle xy\rangle z\rangle$
$\forall x\ c\langle xe\rangle=x$
$\therefore\ \forall x\ c\langle xd\langle x\rangle\rangle=e$

3. Show that the second axiom for groups does not follow from the first two axioms; i.e. that this is invalid:

$\forall x\forall y\forall z\ c\langle xc\langle yz\rangle\rangle=c\langle c\langle xy\rangle z\rangle$
$\forall x\ c\langle xd\langle x\rangle\rangle=e$
$\therefore\ \forall x\ c\langle xe\rangle=x$

4. Show that the first axiom for groups does not follow from the first two axioms; i.e. that this is invalid:

$\forall x\ c\langle xe\rangle=x$
$\forall x\ c\langle xd\langle x\rangle\rangle=e$
$\therefore\ \forall x\forall y\forall z\ c\langle xc\langle yz\rangle\rangle=c\langle c\langle xy\rangle z\rangle$

Answers to Exercises for Chapter Five

1 IDENTITY

1. Say which of the following are formulas:

a.	Fa ∧ Gb ∧ F=G	No	(identity cannot be flanked by predicates)
b.	∀x∀y[R(xy) → x=y]	Yes	
c.	∀x∀y[R(xy) ∧ x≠y ↔ S(yx)]	Yes	
d.	R(xy) ∧ R(yx) ↔ x=y	Yes	
e.	∃x∃y[x=y ∧ y≠x]	Yes	

2. Symbolize the following English sentences:

a. Bruce Wayne is Batman
a=b

b. Bruce Wayne isn't Superman
~a=d or a≠d d: Superman

c. If Clark Kent is Superman, Clark Kent is not from Earth
c=d → ~F(ce)

d. If Clark Kent is Superman, Superman is a reporter
c=d → Ed

e. Felecia chased a dog and Cecelia chased a dog.
∃x(Dx ∧ H(fx)) ∧ ∃x(Dx ∧ H(cx))

f. Felecia chased a dog and Cecelia chased the same dog.
∃x(Dx ∧ H(fx) ∧ H(cx)) or ∃x(Dx ∧ H(fx) ∧ ∃y(Dy ∧ H(cy) ∧ y=x))

g. Felecia chased a dog and Cecelia chased a different dog.
∃x(Dx ∧ H(fx) ∧ ∃y(Dy ∧ H(cy) ∧ y≠x))

2 AT LEAST, AT MOST, EXACTLY, AND ONLY

1. Symbolize each of the following,

a. *At most one candidate will win at least two elections*

∀x∀y(x *is a candidate that wins at least two elections* ∧ y *is a candidate that wins at least two elections* → x=y)

∀x∀y(Cx ∧ x *wins at least two elections* ∧ Cy ∧ y *wins at least two elections* → x=y)

∀x∀y(Cx ∧ ∃z∃u(Ez∧Eu∧z≠u∧W(xz)∧W(xu)) ∧ Cy ∧ ∃z∃u(Ez∧Eu∧z≠u∧W(yz)∧W(yu)) → x=y)

b. *Exactly one election will be won by no candidate*

∃x∀y(y *is an election* ∧ *y is won by no candidate* ↔ y=x)

∃x∀y(Ey ∧ ~∃z(Cz ∧ W(zy)) ↔ y=x) W: *wins*

c. *Betty saw at least two hyenas which each saw at most one giraffe.*

∃x∃y(x≠y ∧ x *is a hyena which saw at most one giraffe* ∧ y *is a hyena which saw at most one giraffe* ∧ *Betty saw* x ∧ *Betty saw* y)

∃x∃y(x≠y ∧ Hx ∧ x *saw at most one giraffe* ∧ Hy ∧ y *saw at most one giraffe* ∧ S(bx) ∧ S(by))

∃x∃y(x≠y ∧ Hx ∧ ∀z∀u(Gz∧Gu∧S(xz)∧S(xu) → z=u) ∧ Hy ∧ ∀z∀u(Gz∧Gu∧S(yz)∧S(yu) → z=u) ∧ S(bx) ∧ S(by))

2. One can symbolize '*Betty saw exactly one dragonfly*' as:

∃x∀y[Dy ∧ S(by) ↔ y=x].

Prove that this sentence is equivalent to one of the symbolizations given above in the text.

1.	~~Show~~ ∃x∀y[Dy ∧ S(by) ↔ y=x] ↔ ∃x[Dx ∧ S(bx) ∧ ∀y[Dy ∧ S(by) → y=x]]	
2.	~~Show~~ ∃x∀y[Dy ∧ S(by) ↔ y=x] → ∃x[Dx ∧ S(bx) ∧ ∀y[Dy ∧ S(by) → y=x]]	
3.	∃x∀y[Dy ∧ S(by) ↔ y=x]	ass cd
4.	∀y[Dy ∧ S(by) ↔ y=u]	3 ei
5.	Du ∧ S(bu) ↔ u=u	4 ui
6.	u=u	sid
7.	Du ∧ S(bu)	5 bc 6 mp
8.	~~Show~~ ∀y[Dy ∧ S(by) → y=u]	
9.	~~Show~~ Dy ∧ S(by) → y=u	
10.	Dy ∧ S(by)	ass cd
11.	Dy ∧ S(by) ↔ y=u	4 ui
12.	y=u	11 bc 10 mp cd
13.		9 ud
14.	Du ∧ S(bu) ∧ ∀y[Dy ∧ S(by) → y=u]	5 8 adj
15.	∃x[Dx ∧ S(bx) ∧ ∀y[Dy ∧ S(by) → y=x]]	14 eg cd
16.	~~Show~~ ∃x[Dx ∧ S(bx) ∧ ∀y[Dy ∧ S(by) → y=x]] → ∃x∀y[Dy ∧ S(by) ↔ y=x]	
17.	∃x[Dx ∧ S(bx) ∧ ∀y[Dy ∧ S(by) → y=x]]	ass cd
18.	Dv ∧ S(bv) ∧ ∀y[Dy ∧ S(by) → y=v]	17 ei
19.	∀y[Dy ∧ S(by) → y=v]	18 s
20.	~~Show~~ ∀y[Dy ∧ S(by) ↔ y=v]	
21.	~~Show~~ Dy ∧ S(by) ↔ y=v	
22.	~~Show~~ y=v → Dy ∧ S(by)	
23.	y=v	ass cd
24.	Dy ∧ S(by)	18 s 23 LL cd
25.	Dy ∧ S(by) → y=v	18 s ui
26.	Dy ∧ S(by) ↔ y=v	22 25 cb dd
27.		21 ud
28.		20 cd
29.	∃x∀y[Dy ∧ S(by) ↔ y=x] ↔ ∃x[Dx ∧ S(bx) ∧ ∀y[Dy ∧ S(by) → y=x]]	2 16 cb dd

3. Similarly show that one can symbolize '*Betty saw exactly two dragonflies*' as:

∃x∃y[x≠y ∧ ∀z[Dz ∧ S(bz) ↔ z=x ∨ z=y]

That is, show

∴ ∃x∃y[x≠y∧∀z[Dz∧S(bz) ↔ z=x∨z=y] ↔ ∃x∃y[Dx∧S(bx)∧Dy∧S(by)∧y≠x∧∀z[Dz∧S(bz) → x=z∨y=z]]

It is straightforward but quite tedious to write out a derivation for this equivalence.

4. Show that the two symbolizations proposed above for *only Cynthia sees Dorothy* are equivalent:

∴ S(cd) ∧ ∀x[S(xd) → x=c] ↔ ∀x[S(xd) ↔ x=c]

1.	~~Show~~ S(cd) ∧ ∀x[S(xd) → x=c] ↔ ∀x[S(xd) ↔ x=c]	
2.	~~Show~~ S(cd) ∧ ∀x[S(xd) → x=c] → ∀x[S(xd) ↔ x=c]	
3.	S(cd) ∧ ∀x[S(xd) → x=c]	ass cd
4.	~~Show~~ ∀x[S(xd) ↔ x=c]	
5.	~~Show~~ S(xd) ↔ x=c	
6.	~~Show~~ S(xd) → x=c	
7.	S(xd)	ass cd
8.	x=c	3 s ui 7 mp cd
9.	~~Show~~ x=c → S(xd)	
10.	x=c	ass cd
11.	S(xd)	3 s 8 LL cd <LL is introduced in Sec 3>
12.	S(xd) ↔ x=c	6 9 cb dd
13.		5 ud
14.		4 cd
15.	~~Show~~ ∀x[S(xd) ↔ x=c] → S(cd) ∧ ∀x[S(xd) → x=c]	
16.	∀x[S(xd) ↔ x=c]	ass cd
17.	S(cd) ↔ c=c	16 ui
18.	S(cd)	sid 17 bp
19.	~~Show~~ ∀x[S(xd) → x=c]	
20.	~~Show~~ S(xd) → x=c	
21.	S(xd) ↔ x=c	16 ui
22.	S(xd) → x=c	21 bc dd
23.		20 ud
24.	S(cd) ∧ ∀x[S(xd) → x=c]	18 19 adj cd
25.	S(cd) ∧ ∀x[S(xd) → x=c] ↔ ∀x[S(xd) ↔ x=c]	2 15 cb dd

3 DERIVATIONAL RULES FOR IDENTITY

1. Derivations are not given here for numbered theorems.

2\. a. ∀x[Fx → x=a ∨ x=b]
~Fa
~Gb
∴ ∀x[Fx → ~Gx]

1.	~~Show~~ ∀x[Fx → ~Gx]	
2.	~~Show~~ Fx → ~Gx	
3.	Fx	ass cd
4.	x≠a	pr2 3 LL ← contrapositive form of LL
5.	Fx → x=a ∨ x=b	pr1 ui
6.	x=a ∨ x=b	3 5 mp
7.	x=b	4 6 mtp
8.	~Gx	7 pr3 LL cd
9.		2 ud

b. ∃x∀y[Ay ↔ y=x]
∴ ∃x[Ax ∧ ~Bx] ↔ ~∃x[Ax ∧ Bx]

1.	~~Show~~ ∃x[Ax ∧ ~Bx] ↔ ~∃x[Ax ∧ Bx]	
2.	~~Show~~ ∃x[Ax ∧ ~Bx] →~∃x[Ax ∧ Bx]	
3.	∃x[Ax ∧ ~Bx]	ass cd
4.	Au ∧ ~Bu	3 ei
5.	~~Show~~ ~∃x[Ax∧Bx]	
6.	∃x[Ax∧Bx]	ass id
7.	Av∧Bv	6 ei
8.	∀y[Ay ↔ y=w]	pr1 ei
9.	Au ↔ u=w	8 ui
10.	u=w	4 2 9 bc mp
11.	Av ↔ v=w	8 ui
12.	v=w	7 s 9 bc mp
13.	u=v	10 12 LL
14.	~Bv	4 s 13 LL
15.	Bv	7 s 14 id
16.		5 cd
17.	~~Show~~ ~∃x[Ax ∧ Bx] → ∃x[Ax ∧ ~Bx]	
18.	~∃x[Ax∧Bx]	ass cd
19.	∀y[Ay ↔ y=i]	pr1 ei
20.	Ai ↔ i=i	19 ui
21.	i=i	sid
22.	Ai	20 bc 21 mp
23.	∀x~[Ax ∧ Bx]	18 qn
24.	~[Ai ∧ Bi]	23 ui
25.	~Ai ∨ ~Bi	24 dm
26.	~Bi	22 dn 25 mtp
27.	Ai ∧ ~Bi	22 26 adj
28.	∃x[Ax∧Bx]	27 eg cd
29.	∃x[Ax ∧ ~Bx] ↔ ~∃x[Ax ∧ Bx]	2 17 bc dd

c. ∃x∃y[x≠y ∧ Gx ∧ Gy]
∀x[Gx → Hx]
∴ ~∃x∀y[Hy ↔ y=x]

Ch5-3.2.c: ∃x∃y(x≠y ∧ Gx ∧ Gy) . ∀x(Gx → Hx) ∴ ~∃x∀y(Hy ↔ y=x)

1	~~Show~~ ~∃x∀y(Hy↔y=x)	"show conc"
2	∃x∀y(Hy↔y=x)	ass id
3	u≠v ∧Gu∧Gv	pr1 ei/u ei/v
4	∀y(Hy↔y=w)	2 ei/w
5	Gu→Hu	pr2 ui/u
6	Hu	3 s s 5 mp
7	Gv →Hv	pr2 ui/v
8	Hv	3 s 7 mp
9	Hu↔u=w	4 ui/u
10	u=w	9 bc 6 mp
11	Hv ↔v=w	4 ui/v
12	v=w	11 bc 8 mp
13	u=v	10 12 LL
14	u≠v	3 s s
15		13 14 id

d. ∃x∃y[Fx ∧ Fy ∧ x≠y]
∃x∃y[Gx ∧ Gy ∧ x≠y]
∴ ∃x∃y[Fx ∧ Gy ∧ x≠y]

Ch5-3.2.d: ∃x∃y(Fx∧Fy∧x≠y) . ∃x∃y(Gx∧Gy∧x≠y) ∴ ∃x∃y(Fx∧Gy∧x≠y)

1	~~Show~~ ∃x∃y(Fx∧Gy∧x≠y)	"show conc"
2	~∃x∃y(Fx∧Gy∧x≠y)	ass id
3	Fu∧Fv ∧u≠v	pr1 ei/u ei/v
4	Gw ∧Gz∧w≠z	pr2 ei/w ei/z
5	∀x~∃y(Fx∧Gy∧x≠y)	2 qn
6	∀x∀y~(Fx∧Gy∧x≠y)	5 ie/qn
7	∀x∀y~~(Fx∧Gy→x=y)	6 ie/nc
8	∀x∀y(Fx∧Gy→x=y)	7 ie/dn
9	Fu∧Gw →u=w	8 ui/u ui/w
10	Fu∧Gw	3 sl sl 4 sl sl adj
11	u=w	9 10 mp
12	Fv ∧Gw	3 sl sr 4 sl sl adj
13	Fv ∧Gw →v=w	8 ui/v ui/w
14	v=w	12 13 mp
15	u=v	11 14 LL
16	u≠v	3 sr
17		15 16 id

3. Symbolize these arguments and produce derivations to show that they are valid.

a. Every giraffe that loves some other giraffe loves itself.
Every giraffe loves some giraffe.
∴ Every giraffe loves itself.

Ch5-3.3.a: ∀x(Gx∧∃y(Gy∧ x≠y∧L(xy)) →L(xx)) . ∀x(Gx→∃y(Gy∧L(xy))) ∴ ∀x(Gx→L(xx))

1	**~~Show~~ ∀x(Gx→L(xx))**	**"show conc"**
2	**~∀x(Gx→L(xx))**	**ass id**
3	**~(Gu→L(uu))**	**2 qn ei/u**
4	**Gu∧~L(uu)**	**3 nc**
5	**Gu→∃y(Gy∧L(uy))**	**pr2 ui/u**
6	**∃y(Gy∧L(uy))**	**4 sl 5 mp**
7	**Gv∧L(uv)**	**6 ei/v**
8	**Gu∧∃y(Gy∧u≠y∧L(uy))→L(uu)**	**pr1 ui/u**
9	**~(Gu∧∃y(Gy∧u≠y∧L(uy)))**	**4 sr 8 mt**
10	**~Gu∨~∃y(Gy∧u≠y∧L(uy))**	**9 dm**
11	**~∃y(Gy∧u≠y∧L(uy))**	**4 sl dn 10 mtp**
12	**∀y~(Gy∧u≠y∧L(uy))**	**11 qn**
13	**~(Gv∧u≠v∧L(uv))**	**12 ui/v**
14	**~(Gv∧u≠v)∨~L(uv)**	**13 dm**
15	**~(Gv∧u≠v)**	**7 sr dn 14 mtp**
16	**~Gv∨~u≠v**	**15 dm**
17	**~u≠v**	**7 sl dn 16 mtp**
18	**u=v**	**17 dn**
19	**L(uv)**	**7 sr**
20	**L(uu)**	**18 19 LL**
21	**~L(uu)**	**4 sr**
22		**20 21 id**

b. No cat that likes at least two dogs is happy.
Tabby is a cat that likes Fido.
Tabby likes a dog that Betty owns.
Fido is a dog.
Tabby is happy.
∴ Betty owns Fido.

~∃x[Cx ∧ ∃y∃z[Dy ∧ Dz ∧ y≠z ∧ L(xy) ∧ L(xz)] ∧ Hx]
Ca ∧ L(af)
∃x[Dx ∧ O(bx) ∧ L(ax)]
Df
Ha
∴ O(bf)

Ch5-3-3-b: ~∃x(Cx∧∃y∃z(Dy∧Dz∧y≠z∧L(xy)∧L(xz))∧Hx) . Ca∧L(af) . ∃x(Dx∧O(bx)∧L(ax)) . Df . Ha ∴ O(bf)

1	~~Show~~ O(bf)	"show conc"
2	~O(bf)	ass id
3	Du∧O(bu)∧L(au)	pr3 ei/u
4	O(bu)	3 sl sr
5	~f=u	2 4 LL
6	Du	3 sl sl
7	Df∧Du∧~f=u	pr4 6 adj 5 adj
8	Df∧Du∧~f=u∧L(af)	pr2 sr 7 adj
9	Df∧Du∧~f=u∧L(af)∧L(au)	3 sr 8 adj
10	∃z(Df∧Dz∧~f=z∧L(af)∧L(az))	9 eg/z
11	∃y∃z(Dy∧Dz∧~y=z∧L(ay)∧L(az))	10 eg/y
12	Ca∧∃y∃z(Dy∧Dz∧~y=z∧L(ay)∧L(az))	pr2 sl 11 adj
13	Ca∧∃y∃z(Dy∧Dz∧~y=z∧L(ay)∧L(az))∧Ha	12 pr5 adj
14	∃x(Cx∧∃y∃z(Dy∧Dz∧~y=z∧L(xy)∧L(xz))∧Hx)	13 eg/x
15	~∃x(Cx∧∃y∃z(Dy∧Dz∧y≠z∧L(xy)∧L(xz))∧Hx)	pr1
16		14 15 id

c. Each widget fits into a socket. ∀x[Ix → ∃y[Ey ∧ F(xy)]]
widget a doesn't fit into socket f Ia ∧ Ef ∧ ~F(af)
∴ widget a fits into some socket other than f ∴ Ia ∧ ∃x[Ex ∧ x≠f ∧ F(ax)]

Ch5-3-3-c: ∀x(Ix→∃y(Ey∧F(xy))) . Ia∧Ef∧~F(af) ∴ Ia ∧ ∃x(Ex∧x≠f∧F(ax))

1	~~Show~~ Ia∧∃x(Ex∧x≠f∧F(ax))	"show conc"
2	Ia→∃y(Ey∧F(ay))	pr1 ui/a
3	∃y(Ey∧F(ay))	pr2 sl sl 2 mp
4	Eu∧F(au)	3 ei/u
5	F(au)	4 sr
6	~F(af)	pr2 sr
7	~u=f	5 6 LL
8	Eu∧~u=f	4 sl 7 adj
9	Eu∧~u=f∧F(au)	8 5 adj
10	∃x(Ex∧~x=f∧F(ax))	9 eg/x
11	Ia∧∃x(Ex∧~x=f∧F(ax))	pr2 sl sl 10 adj
12		11 dd

d. Only Betty and Carl were eligible ∀x[Ex ↔ x=b ∨ x=c]
Somebody who was eligible, won ∃x[Ex ∧ Ix]
Carl didn't win ~Ic
∴ Betty won ∴ Ib

1. ~~Show~~ Ib

2.	Eu∧Iu	pr2 ei	
3.	Eu ↔ u=b ∨ u=c	pr1 ui	
4.	u=b ∨ u=c	2 s 3 bp	
5.	~u=c	2 s pr3 LL	<contrapositive form of LL>
6.	u=b	4 5 mpt	
7.	Ib	2 s 6 LL dd	

4 INVALIDITIES WITH IDENTITY

1. Only Betty and Carl were eligible
Nobody who wasn't eligible won
Carl didn't win
∴ Betty won

$\forall x[Ex \leftrightarrow x{=}b \vee x{=}c]$	Universe: {0, 1, 2}
$\sim\exists x[\sim Ex \wedge Ix]$	b: 0
$\sim Ic$	c: 1
∴ Ib	E: {0,1}
	I: {}

2. Ann loves at least one freshman.
Ann loves David.
Ed is a freshman.
David isn't Ed.
∴ There are at least two freshmen

$\exists x[Fx \wedge L(ax)]$	Universe: {0, 1, 2}
$L(ad)$	a: 0
Fe	d: 1
$d \neq e$	e: 2
∴ $\exists x\exists y[Fx \wedge Fy \wedge x \neq y]$	F: {2}
	L: {<0,1>, <0,2>}

3. Lois sees Clark at a time if and only if she sees Superman at that time.
∴ Clark is Superman

$\forall x[Tx \rightarrow [S(icx) \leftrightarrow S(iex)]]$ i: Lois c: Clark e: Superman   S(xyz) x sees y at z
∴ $c{=}e$

Universe: {0, 1, 2, 3, 4, 5}   <4 and 5 could be omitted>
i: 0
c: 1
e: 2
T: {3, 4, 5}
S: {<0,1,3>, <0,2,3>, <0,1,4>, <0,2,4>}

4. Gertrude sees at most one giraffe
Gertrude sees Fred, who is a giraffe
Bob is a giraffe
∴ Gertrude doesn't see Bob

$\forall x\forall y(Gx \wedge Gy \wedge S(gx) \wedge S(gy) \rightarrow x{=}y)$	Universe: {0, 1}
$S(gf) \wedge Gf$	g: 0
Gb	b: 1
∴ $\sim S(gb)$	f: 1
	G{1}
	S{<0,1>}

5 OPERATION SYMBOLS

Which of the following are formulas?

a. R(f‹x›g‹x›)	Yes
b. ∀x[Fx → Fg‹x›]	Yes
c. ∀x[Fx → Fg‹xx›]	Yes
d. ∀x∀y[x=h‹y› → f‹xy›=f‹yx›]	Yes
e. ~∃y∃x[x=f‹y› ∧ y=f‹x›]	Yes
f. ~∃x∃yf‹xy›	No. There is no predicate letter.
g. S(xyz) ∨ ~S(xg‹y›z) ∨ ~S(g‹xy›g‹z›g‹yz›)	Yes
h. Fa ∧ ~Fb → [Fg‹a› → g‹b›≠g‹a›]	Yes

6 DERIVATIONS WITH COMPLEX TERMS

1. Give derivations for these theorems:

a. ∴ ∀xR(xf‹x›) → ∀xR(e‹x›f‹e‹x››)

1. ~~Show~~ ∀xR(xf‹x›) → ∀xR(e‹x›f‹e‹x››)
2. ∀xR(xf‹x›)
3. ~~Show~~ ∀xR(e‹x›f‹e‹x››)
4. ~~Show~~ R(e‹x›f‹e‹x››)
5. R(e‹x›f‹e‹x››) 2 ui dd
6. 4 ud
7. 3 cd

b. ∴ ∀x∀y[x=f‹y› ∧ y=f‹x› → f‹f‹x››=x]

1. ~~Show~~ ∀x∀y[x=f‹y› ∧ y=f‹x› → f‹f‹x››=x]

2.	~~Show~~ x=f‹y› ∧ y=f‹x› → f‹f‹x››=x	
3.	x=f‹y› ∧ y=f‹x›	ass cd
4.	x= f‹y›	3 s
5.	y=f‹x›	3 s
6.	x = f‹f‹x››	4 5 LL
7.	f‹f‹x›› = x	6 sm cd
8.		2 ud

2. Show that these are consequences of the theory of biological kinship given above.

a. ~∃x[∃zB(xz) ∧ ∃zD(xz)] *No brother is a daughter*

P1	∀xAf‹x›	*Everyone's father is male*
P2	∀xEe‹x›	*Everyone's mother is female*
P3	∀x∀y[I(xy) ↔ x≠y∧f‹x›=f‹y›∧e‹x›=e‹y›]	*(Full) Siblings have the same mother and father*
P4	∀x∀y[B(xy) ↔ Ax ∧ I(xy)]	*A brother of someone is his/her male sibling*
P5	∀x∀y[D(xy) ↔ Ex ∧ [y=f‹x› ∨ y=e‹x›]]	*A daughter of a person is a female such that that person is her father or her mother*
P6	∀x[Ax ↔ ~Ex]	*Someone is male if and only if that person is not female*

1. ~~Show~~ ~∃x[∃zB(xz) ∧ ∃zD(xz)]

2.	∃x[∃zB(xz) ∧ ∃zD(xz)]	ass id
3.	∃zB(uz) ∧ ∃zD(uz)	2 ei
4.	B(uv)	3 s ei
5.	D(uw)	3 s ei
6.	B(uv) ↔ Au ∧ I(uv)	pr4 ui ui
7.	Au	6 bc 4 mp s
8.	D(uw) ↔ Eu ∧ [w=f‹u› ∨ w=e‹u›]	pr5 ui ui
9.	Eu	8 bc 5 mp s
10.	Au ↔ ~Eu	pr6 ui
11.	~Eu	10 bc 7 mp
12.		9 11 id

b. ~∃x[∃z x=f‹z› ∧ ∃z x=e‹z›] *No father is a mother*

P1	∀xAf‹x›	*Everyone's father is male*
P2	∀xEe‹x›	*Everyone's mother is female*
P3	∀x∀y[I(xy) ↔ x≠y∧f‹x›=f‹y›∧e‹x›=e‹y›]	*(Full) Siblings have the same mother and father*
P4	∀x∀y[B(xy) ↔ Ax ∧ I(xy)]	*A brother of someone is his/her male sibling*
P5	∀x∀y[D(xy) ↔ Ex ∧ [y=f‹x› ∨ y=e‹x›]]	*A daughter of a person is a female such that that person is her father or her mother*
P6	∀x[Ax ↔ ~Ex]	*Someone is male if and only if that person is not female*

1. ~~Show~~ ~∃x[∃z x=f‹z› ∧ ∃z x=e‹z›]

2.	∃x[∃z x=f‹z› ∧ ∃z x=e‹z›]	ass id
3.	∃z u=f‹z› ∧ ∃z u=e‹z›	2 ei
4.	u=f‹v›	3 s ei
5.	u=e‹w›	3 s ei
6.	Af‹v›	pr1 ui
7.	Au	4 6 LL
8.	Ee‹w›	pr2 ui
9.	Eu	5 8 LL
10.	Au ↔ ~Eu	pr6 ui
11.	~Eu	10 bc 7 mp
12.		9 11 id

3. Show that these are consequences of the axioms for groups.

Group Theorem a. ∀x∀y∀z[c‹xy›=c‹zy› → x=z]

<<Given in the text, using different bound variables>>

Group Theorem b. ∀x[∀yc‹yx›=y → x=e]

∀x∀y∀z c‹xc‹yz››=c‹c‹xy›z›
∀x c‹xe›=x
∀x c‹xd‹x››=e
∴ ∀x[∀yc‹yx›=y → x=e]

1. ~~Show~~ ∀x[∀yc‹yx›=y → x=e]

2.	~~Show~~ ∀yc‹yx›=y → x=e	
3.	∀yc‹yx›=y	ass cd
4.	c‹xx›=x	3 ui
5.	c‹ex›=e	3 ui
6.	c‹ed‹x››=c‹ed‹x››	sid
7.	c‹c‹ex›d‹x››=c‹ed‹x››	5 6 LL
8.	c‹c‹ec‹xx››d‹x››=c‹ed‹x››	4 7 LL
9.	c‹c‹c‹ex›x›d‹x››=c‹ed‹x››	pr1 ui ui ui 8 LL
10.	c‹c‹ex›d‹x››=c‹ed‹x››	5 9 LL
11.	c‹ec‹xd‹x›››=c‹ed‹x››	pr1 ui ui ui 10 LL
12.	c‹ee›=c‹ed‹x››	pr3 ui 11 LL
13.	e=c‹ed‹x››	pr2 ui 12 LL
14.	c‹xd‹x››=c‹ed‹x››	pr3 ui 13 LL
15.	c‹xd‹x››=c‹ed‹x›› → x=e	Group Theorem a ui ui ui
16.	x=e	14 15 mp cd
17.		2 ud

Group Theorem c. ∀x c‹xd‹x››=c‹d‹x›x›

∀x∀y∀z c‹xc‹yz››=c‹c‹xy›z›
∀x c‹xe›=x
∀x c‹xd‹x››=e
∴ ∀x c‹xd‹x››=c‹d‹x›x›

1. ~~Show~~ ∀x c‹xd‹x››=c‹d‹x›x›

2.	~~Show~~ c‹xd‹x››=c‹d‹x›x›	
3.	~~Show~~ ∀y c(yc‹d‹x›x›) = y	
4.	~~Show~~ c(yc‹d‹x›x›) = y	
5.	c‹yd‹x›› = c‹yd‹x››	sid
6.	c‹yc‹d‹x›e› = c‹yd‹x››	pr2 ui 5 LL
7.	c‹yc‹d‹x›c‹xd‹x››› = c‹yd‹x››	pr3 ui 6 LL
8.	c‹yc‹c‹d‹x›x›d‹x››› = c‹yd‹x››	pr1 ui ui ui 7 LL
9.	c‹c‹yc‹d‹x›x››d‹x›› = c‹yd‹x››	pr1 ui ui ui 8 LL
10.	c‹yc‹d‹x›x›› = y	Group Theorem a ui ui ui 9 mp
11.		10 dd
12.		4 ud
13.	∀yc‹y c‹d‹x›x››=y → c‹d‹x›x›=e	Group Theorem b ui
14.	c‹d‹x›x› = e	13 3 mp
15.	c‹xd‹x›› = e	pr3 ui
16.	c‹xd‹x›› = c‹d‹x›x›	14 15 LL
17.		16 dd
18.		2 ud

Group Theorem d. ∀x∀y∀z[c‹yx›=c‹yz› → x=z]

1. ~~Show~~ ∀x∀y∀z[c‹yx›=c‹yz› → x=z]

2.	~~Show~~ c‹yx›=c‹yz› → x=z	
3.	c‹yx›=c‹yz›	ass cd
4.	c‹d‹y›c‹yx›› = c‹d‹y›c‹yz››	3 EL
5.	c‹c‹d‹y›y›x› = c‹c‹d‹y›y›z›	pr1 ui ui ui 4 LL
6.	c‹c‹yd‹y››x› = c‹c‹yd‹y››z›	Group theorem c ui 5 LL
7.	c‹c‹yd‹y››x› = c‹ez›	pr2 ui 6 LL
8.	c‹ex›= c‹ez›	pr2 ui 7 LL
9.	c‹c‹xd‹x››x› = c‹ez›	pr2 ui 8 LL
10.	c‹c‹xd‹x››x› = c‹zc‹zd‹z››z›	pr1 ui ui ui 9 LL
11.	c‹xc‹d‹x››x› = c‹zc‹d‹z››z›	pr1 ui ui ui 10 LL
12.	c‹xc‹xd‹x››› = c‹zc‹d‹z››z›	Group theorem c ui 11 LL
13.	c‹xc‹xd‹x››› = c‹zc‹zd‹z›››	Group theorem c ui 12 LL
14.	c‹xe› = c‹zc‹zd‹z›››	pr3 ui 13 LL
15.	c‹xe› = c‹ze›	pr3 ui 14 LL
16.	x = c‹ze›	pr2 ui 15 LL
17.	x = z	pr2 ui 16 LL cd
18		2 ud

Group Theorem e. ∀x∀y[c‹xy›=e → y=d‹x›]

∀x∀y∀z c‹xc‹yz››=c‹c‹xy›z›
∀x c‹xe›=x
∀x c‹xd‹x››=e
∴ ∀x∀y[c‹xy›=e → y=d‹x›]

1.	~~Show~~ ∀x∀y[c‹xy›=e → y=d‹x›]	
2.	~~Show~~ c‹xy›=e → y=d‹x›	
3.	c‹xy›=e	ass cd
4.	c‹d‹x›c‹xy››=c‹d‹x›e›	3 EL
5.	c‹c‹d‹x›x›y›=c‹d‹x›e›	pr1 ui ui ui 4 LL
6.	c‹c‹xd‹x›y›= c‹d‹x›e›	Group theorem c 5 LL
7.	c‹ey› = c‹d‹x›e›	pr2 ui 6 LL
8.	c‹ey› = c‹d‹x›c‹xd‹x›››	pr2 ui 7 LL
9.	c‹ey› = c‹c‹d‹x›x›d‹x››	pr1 ui ui ui 8 LL
10.	c‹ey› = c‹c‹xd‹x››d‹x››	Group theorem c ui 9 LL
11.	c‹ey› = c‹ed‹x››	pr3 ui 10 LL
12.	c‹ey› = c‹ed‹x››→y=d‹x›	Group theorem d ui ui ui
13.	y=d‹x›	11 12 mp cd
14.		2 ud ud

Group Theorem f. ∀x d‹d‹x››=x

∀x∀y∀z c‹xc‹yz››=c‹c‹xy›z›
∀x c‹xe›=x
∀x c‹xd‹x››=e
∴ ∀x d‹d‹x››=x

1.	~~Show~~ ∀x d‹d‹x››=x	
2.	~~Show~~ d‹d‹x››=x	
3.	c‹xd‹x››=e	pr3
4.	e= c‹xd‹x››	3 sm
5.	c‹d‹d‹x››d‹x››=e	pr3 ui
6.	c‹d‹d‹x››d‹x››= c‹xd‹x››	4 5 LL
7.	c‹d‹d‹x››d‹x››= c‹xd‹x››→ d‹d‹x››=x	Group theorem a ui ui ui
8.	d‹d‹x››=x	6 7 mp dd
9.		2 ud

7 INVALID ARGUMENTS WITH OPERATION SYMBOLS

1. Produce counter-examples to show these arguments to be invalid:

a. $\forall x \exists y\ a\langle xy \rangle = c$
$\forall x \exists y\ a\langle yx \rangle = c$
$\therefore\ \forall x \forall y\ a\langle xy \rangle = a\langle yx \rangle$

Universe: {0,1}
c: 0
$a\langle 00 \rangle \Rightarrow 0$ $a\langle 11 \rangle \Rightarrow 0$ $a\langle 01 \rangle \Rightarrow 0$ $a\langle 10 \rangle \Rightarrow 1$

b. $\forall x \exists y\ a\langle xy \rangle = c$
$\forall x \exists y\ a\langle xy \rangle = d$
$\therefore\ \exists x \forall y\ a\langle xc \rangle = y$

Universe: {01)
c: 0
d: 1
$a\langle 00 \rangle \Rightarrow 0$ $a\langle 11 \rangle \Rightarrow 1$ $a\langle 01 \rangle \Rightarrow 1$ $a\langle 10 \rangle \Rightarrow 0$

c. $\forall x \forall y\ a\langle xy \rangle = a\langle yx \rangle$
$\therefore\ \forall z \exists x \exists y\ a\langle xy \rangle = z$

Universe: {0,1}
$a\langle 00 \rangle \Rightarrow 0$ $a\langle 11 \rangle \Rightarrow 0$ $a\langle 01 \rangle \Rightarrow 0$ $a\langle 10 \rangle \Rightarrow 0$

2. Show that these are not theorems of the theory of biological kinship given in the previous section:

a. $\forall x[\exists y\ x=f\langle y \rangle \vee \exists y\ x=e\langle y \rangle]$ *Everyone is a father or a mother*

P1	$\forall x A f\langle x \rangle$	*Everyone's father is male*
P2	$\forall x E e\langle x \rangle$	*Everyone's mother is female*
P3	$\forall x \forall y[I(xy) \leftrightarrow x \neq y \wedge f\langle x \rangle = f\langle y \rangle \wedge e\langle x \rangle = e\langle y \rangle]$	*(Full) Siblings have the same mother and father*
P4	$\forall x \forall y[B(xy) \leftrightarrow Ax \wedge I(xy)]$	*A brother of someone is his/her male sibling*
P5	$\forall x \forall y[D(xy) \leftrightarrow Ex \wedge [y=f\langle x \rangle \vee y=e\langle x \rangle]]$	*A daughter of a person is a female such that that person is her father or her mother*
P6	$\forall x[Ax \leftrightarrow \sim Ex]$	*Someone is male if and only if that person is not female*

Universe: (0,1, 2)
A: {0,2}
E: {1}
$f\langle 0 \rangle \Rightarrow 0$ $f\langle 1 \rangle \Rightarrow 0$ $f\langle 2 \rangle \Rightarrow 0$
$e\langle 0 \rangle \Rightarrow 1$ $e\langle 1 \rangle \Rightarrow 1$ $e\langle 2 \rangle \Rightarrow 1$

b. $\sim \exists x\ x=e\langle x \rangle$ *Nobody is their own mother*
[so certain science fiction stories are not ruled out]

The same interpretation will work here:
Universe: (0,1, 2)
A: {0,2}
E: {1}
$f\langle 0 \rangle \Rightarrow 0$ $f\langle 1 \rangle \Rightarrow 0$ $f\langle 2 \rangle \Rightarrow 0$
$e\langle 0 \rangle \Rightarrow 1$ $e\langle 1 \rangle \Rightarrow 1$ $e\langle 2 \rangle \Rightarrow 1$

8 COUNTER-EXAMPLES WITH INFINITE DOMAINS

1. Show that this argument is invalid:

	∀x∀y∀z[R(xy)∧R(yz) → R(xz)]	Universe: {0, 1, 2, . . .}
	∀xR(xf‹x›)	R(①②): ①<②
∴	∃xR(xa)	f‹①›: ①+1
		a: 0

The first premise says that less than is transitive. The second says that each integer is less than the integer you get by adding 1 to it. The conclusion says falsely that there is something in the universe that is less than zero.

2. Show that the third axiom for groups does not follow from the first two axioms; i.e. that this is invalid:

	∀x∀y∀z c‹xc‹yz››=c‹c‹xy›z›	Universe: {0, 1, 2, . . .}
	∀x c‹xe›=x	c‹①②›: ①+②
∴	∀x c‹xd‹x››=e	e: 0
		d‹①›: ①+①

The first premise says truly that addition is commutative. The second says truly that adding 0 to a number yields that number itself. The conclusion says falsely that ①+(①+①)=0 for every integer in the domain.

3. Show that the second axiom for groups does not follow from the first two axioms; i.e. that this is invalid:

	∀x∀y∀z c‹xc‹yz››=c‹c‹xy›z›	Universe: {0, 1, 2, . . .}	
	∀x c‹xd‹x››=e	c‹①②›: ①·②	<multiplication>
∴	∀x c‹xe›=x	d‹①›: ①-①	
		e: 0	

The first premise says truly that multiplication is associative. The second says truly that multiplying any integer by 0 yields 0. The conclusion says falsely that multiplying any integer by zero (by what you get when you subtract that integer from itself) yields that integer.

4. Show that the first axiom for groups does not follow from the first two axioms; i.e. that this is invalid:

	∀x c‹xe›=x	Universe: {. . . , -2, -1, 0, 1, 2, . . .}
	∀x c‹xd‹x››=e	c‹①②›: ①-②
∴	∀x∀y∀z c‹xc‹yz››=c‹c‹xy›z›	d‹①›: ①
		e: 0

The first premise says truly that subtracting zero from any integer yields that integer. The second premise says truly that subtracting any integer from itself yields zero. The conclusion says falsely that subtraction is associative. It's not associative; for example, (5-2)-1=2 whereas 5-(2-1)=4.